MATH4230 Optimization Theory (2017/18)
Tutorial 3

Please do the star problem ($*$) in tutorial class and finish the rest after class. Please hand in your answer sheet to the assignment box in Lady Shaw Building before 6:30 p.m.

1*. Let X be a nonempty convex subset of $\Re^n$, let $f : X \to \Re^n$ be a concave function, and let X^* be the set of vectors where f attains a minimum over X, i.e.,

$$X^* = \{x^* \in X | f(x^*) = inf_{x \in X} f(x)\}.$$

Show that if there exist $x_0 \in X^*$, $x_0 \in ri(X)$, then $f \equiv C$ on X, where C is a constant.

Solution 1. *Let $x^* \in X^* \bigcap ri(X)$, and let x be any vector in X. By the Prolongation Lemma, there exists a $\gamma \geq 0$ s.t.*

$$\hat{x} = x^* + \gamma(x^* - x) \in X.$$

Hence,

$$x^* = \frac{1}{\gamma + 1}\hat{x} + \frac{\gamma}{\gamma + 1}x.$$

By the concavity of f, we have

$$f(x^*) \geqslant \frac{1}{\gamma + 1} f(\hat{x}) + \frac{\gamma}{\gamma + 1} f(x),$$

and using $f(\hat{x}) \geqslant f(x^*), f(x) \geqslant f(x^*)$, this shows that $f(x) = f(x^*)$. □

2*. Let C be a nonempty convex set. Show that

(a) $cl(C) = cl(ri(C))$.

(b) $ri(C) = ri(cl(C))$.

(c) Let $\bar{C}$ be another nonempty convex set. Then the following three conditions are equivalent:

(i) C and $\bar{C}$ have the same relative interior.

(ii) C and $\bar{C}$ have the same closure.

(iii) $ri(C) \subset \bar{C} \subset cl(C)$.

Solution 2. (*a*) *Since $ri(C) \subset C$, we have $cl(ri(C)) \subset cl(C)$. Conversely, let $\hat{x} \in cl(C)$. We will show that $\hat{x} \in cl(ri(C))$. Let $x \in ri(C)$ be any point(the existence is ensured by Noemptiness of Relative Interior), and assume $\hat{x} \neq x$ (otherwise we are done). By the Line Segment Principle, we have*

$$\alpha x + (1 - \alpha)\hat{x} \in ri(C), \forall \alpha \in (0, 1].$$

Thus, $\hat{x}$ is the limit of the sequence

$$\{x_k = \frac{1}{k}x + (1 - \frac{1}{k})\hat{x} | k = 1, 2, 3 \dots\}$$

that lies in $ri(C)$, *so* $\hat{x} \in cl(ri(C))$.

(*b*) *The inclusion* $ri(C) \subset ri(cl(C))$ *follows from the definition of a relative interior point and the fact* $aff(C) = aff(cl(C))$ *(the proof of this is left for the reader). To prove the reverse inclusion, let* $z \in ri(cl(C))$. *We will show that* $z \in ri(C)$. *There exists an* $x \in ri(C)$. *We may assume that* $x \neq z$ *(Otherwise we are done). We use the Prolongation Lemma to choose* $\gamma \geq 0$, *with* γ *sufficiently close to* 0 *so that the vector*

$$y = z + \gamma(z - x) \in cl(C).$$

Then we have

$$z = (1 - \alpha)x + \alpha y$$

where $\alpha = \frac{1}{\gamma+1} \in (0, 1)$, *so by the Line Segment Principle (applied within the set* C*), we obtain* $z \in ri(C)$.

(*c*) *We prove this equivalence argument by the following steps:*
$i \rightarrow ii$: *part (a).*
$ii \rightarrow i$: *part (b).*
$i, ii \rightarrow iii$ *obviously.*
$iii \rightarrow ii$ *Supppose the condition iii holds. Then by taking closures, we have* $cl(ri(C)) \subset (\bar{C}) \subset cl(C)$, *and by using part (a), we obtain* $cl(C) \subset cl(\bar{C}) \subset cl(C)$. *Hence* $cl(\bar{C}) = cl(C)$. □

3. Let X be a nonempty set. Show that:

   (a) X, conv(X), and cl(X) have the same affine hull.

   (b) $cone(X) = cone(conv(X))$.

   (c) aff(conv(X)) $\subset$aff(cone(X)). Give an example where the inclusion is strict, i.e., aff(conv(X)) is a strict subset of aff(cone(X)).

   (d) If the origin belongs to conv(X), then aff(conv(X)) =aff(cone(X)).

Solution 3.

(a) We first show that X and $\mathrm{cl}(X)$ have the same affine hull. Since $X \subset \mathrm{cl}(X)$, there holds

$$\mathrm{aff}(X) \subset \mathrm{aff}\big(\mathrm{cl}(X)\big).$$

Conversely, because $X \subset \mathrm{aff}(X)$ and $\mathrm{aff}(X)$ is closed, we have $\mathrm{cl}(X) \subset \mathrm{aff}(X)$, implying that

$$\mathrm{aff}\big(\mathrm{cl}(X)\big) \subset \mathrm{aff}(X).$$

We now show that X and $\mathrm{conv}(X)$ have the same affine hull. By using a translation argument if necessary, we assume without loss of generality that X contains the origin, so that both $\mathrm{aff}(X)$ and $\mathrm{aff}\big(\mathrm{conv}(X)\big)$ are subspaces. Since $X \subset \mathrm{conv}(X)$, evidently $\mathrm{aff}(X) \subset \mathrm{aff}\big(\mathrm{conv}(X)\big)$. To show the reverse inclusion, let the dimension of $\mathrm{aff}\big(\mathrm{conv}(X)\big)$ be m, and let $x_1, \ldots, x_m$ be linearly independent vectors in $\mathrm{conv}(X)$ that span $\mathrm{aff}\big(\mathrm{conv}(X)\big)$. Then every $x \in \mathrm{aff}\big(\mathrm{conv}(X)\big)$ is a linear combination of the vectors $x_1, \ldots, x_m$, i.e., there exist scalars $\beta_1, \ldots, \beta_m$ such that

$$x = \sum_{i=1}^{m} \beta_i x_i.$$

By the definition of convex hull, each x_i is a convex combination of vectors in X, so that x is a linear combination of vectors in X, implying that $x \in \mathrm{aff}(X)$. Hence, $\mathrm{aff}\big(\mathrm{conv}(X)\big) \subset \mathrm{aff}(X)$.

(b) Since $X \subset \mathrm{conv}(X)$, clearly $\mathrm{cone}(X) \subset \mathrm{cone}\big(\mathrm{conv}(X)\big)$. Conversely, let $x \in \mathrm{cone}\big(\mathrm{conv}(X)\big)$. Then x is a nonnegative combination of some vectors in $\mathrm{conv}(X)$, i.e., for some positive integer p, vectors $x_1, \ldots, x_p \in \mathrm{conv}(X)$, and nonnegative scalars $\alpha_1, \ldots, \alpha_p$, we have

$$x = \sum_{i=1}^{p} \alpha_i x_i.$$

Each x_i is a convex combination of some vectors in X, so that x is a nonnegative combination of some vectors in X, implying that $x \in \mathrm{cone}(X)$. Hence $\mathrm{cone}\big(\mathrm{conv}(X)\big) \subset \mathrm{cone}(X)$.

(c) Since $\mathrm{conv}(X)$ is the set of all convex combinations of vectors in X, and $\mathrm{cone}(X)$ is the set of all nonnegative combinations of vectors in X, it follows that $\mathrm{conv}(X) \subset \mathrm{cone}(X)$. Therefore

$$\mathrm{aff}\big(\mathrm{conv}(X)\big) \subset \mathrm{aff}\big(\mathrm{cone}(X)\big).$$

As an example showing that the above inclusion can be strict, consider the set $X = \big\{(1,1)\big\}$ in $\Re^2$. Then $\mathrm{conv}(X) = X$, so that

$$\mathrm{aff}\big(\mathrm{conv}(X)\big) = X = \big\{(1,1)\big\},$$

and the dimension of $\mathrm{conv}(X)$ is zero. On the other hand, $\mathrm{cone}(X) = \big\{(\alpha, \alpha) \mid \alpha \geq 0\big\}$, so that

$$\mathrm{aff}\big(\mathrm{cone}(X)\big) = \big\{(x_1, x_2) \mid x_1 = x_2\big\},$$

and the dimension of cone(X) is one.

(d) In view of parts (a) and (c), it suffices to show that

$$\mathrm{aff}\big(\mathrm{cone}(X)\big) \subset \mathrm{aff}\big(\mathrm{conv}(X)\big) = \mathrm{aff}(X).$$

It is always true that $0 \in \mathrm{cone}(X)$, so $\mathrm{aff}\big(\mathrm{cone}(X)\big)$ is a subspace. Let the dimension of $\mathrm{aff}\big(\mathrm{cone}(X)\big)$ be m, and let $x_1, \ldots, x_m$ be linearly independent vectors in cone(X) that span $\mathrm{aff}\big(\mathrm{cone}(X)\big)$. Since every vector in $\mathrm{aff}\big(\mathrm{cone}(X)\big)$ is a linear combination of $x_1, \ldots, x_m$, and since each x_i is a nonnegative combination of some vectors in X, it follows that every vector in $\mathrm{aff}\big(\mathrm{cone}(X)\big)$ is a linear combination of some vectors in X. In view of the assumption that $0 \in \mathrm{conv}(X)$, the affine hull of conv(X) is a subspace, which implies by part (a) that the affine hull of X is a subspace. Hence, every vector in $\mathrm{aff}\big(\mathrm{cone}(X)\big)$ belongs to $\mathrm{aff}(X)$, showing that $\mathrm{aff}\big(\mathrm{cone}(X)\big) \subset \mathrm{aff}(X)$.